

THE n -TH REDUCED BKP HIERARCHY, THE STRING EQUATION AND $BW_{1+\infty}$ -CONSTRAINTS

JOHAN VAN DE LEUR*

Abstract

We study the BKP hierarchy and its n -reduction, for the case that n is odd. This is related to the principal realization of the basic module of the twisted affine Lie algebra $\hat{sl}_n^{(2)}$. We show that the following two statements for a BKP τ function are equivalent: (1) τ is n -reduced and satisfies the string equation, i.e. $L_{-1}\tau = 0$, where L_{-1} is an element of some ‘natural’ Virasoro algebra. (2) τ satisfies the vacuum constraints of the $BW_{1+\infty}$ algebra. Here $BW_{1+\infty}$ is the natural analog of the $W_{1+\infty}$ algebra, which plays a role in the KP case.

1. Introduction

1.1. In recent years KdV type hierarchies have been related to 2D gravity. To be slightly more precise (see [Dij] for the details and references), the square root of the partition function of the Hermitian $(n-1)$ -matrix model in the continuum limit is the τ -function of the n -reduced Kadomtsev Petviashvili (KP) hierarchy. The partition function is then characterized by the so-called string equation:

$$L_{-1}\tau = \frac{1}{n} \frac{\partial \tau}{\partial x_1},$$

where L_{-1} is an element of the $c = n$ Virasoro algebra, which is related to the principal realization of the affine Lie algebra \hat{sl}_n , or rather \hat{gl}_n . Let $\alpha_k = -kx_{-k}$, 0 , $\frac{\partial}{\partial x_k}$ for $k < 0$, $k = 0$, $k > 0$, respectively, then

$$L_k = \frac{1}{2n} \sum_{\ell \in \mathbf{Z}} : \alpha_{-\ell} \alpha_{\ell+nk} : + \delta_{0k} \frac{n^2 - 1}{24n}. \quad (1.1)$$

By making the shift $x_{n+1} \mapsto x_{n+1} + \frac{n}{n+1}$, we modify the origin of the τ -function and thus obtain the following form of the string equation:

$$L_{-1}\tau = 0. \quad (1.2)$$

Actually, it can be shown ([FKN], [G] and [AV]) that the above conditions, n -th reduced KP and equation (1.2) (which from now on we will call the string equation), on a τ -function of the KP hierarchy imply more general constraints, viz. the vacuum constraints of the $W_{1+\infty}$ algebra. This last condition is reduced to the vacuum conditions of the W_n algebra when some redundant variables are eliminated.

The $W_{1+\infty}$ algebra is the central extension of the Lie algebra of differential operators on \mathbf{C}^\times . This central extension was discovered by Kac and Peterson in 1981 [KP] (see also [R], [KR]). It has as basis the operators $W_k^{(\ell+1)} = -s^{k+\ell}(\frac{\partial}{\partial s})^\ell$, $\ell \in \mathbf{Z}_+$, $k \in \mathbf{Z}$, together with the central element c . There is a well-known

* The research of Johan van de Leur is financially supported by the “Stichting Fundamenteel Onderzoek der Materie (F.O.M.)”. E-mail: vdleur@math.utwente.nl

way how to express these elements in the elements of the Heisenberg algebra, the α_k 's. The $W_{1+\infty}$ constraints then are

$$\hat{W}_k^{(\ell+1)}\tau = \{W_k^{(\ell+1)} + \delta_{k,0}c_{\ell+1}\}\tau = 0 \quad \text{for } -k \leq \ell \text{ and } \ell \geq 0.$$

For the above τ -function, $\hat{W}_k^{(1)} = -\alpha_{nk}$ and $\hat{W}_k^{(2)} = L_k - \frac{nk+1}{n}\alpha_{nk}$.

1.2. In this paper we study the n -th reduced BKP hierarchy, where we assume that n is odd. This reduction is related to the principal realization of the basic module of the affine Lie algebra $\hat{sl}_n^{(2)}$. A τ -function of the n -th reduced BKP hierarchy is a function in the variables x_1, x_3, x_5, \dots with the restriction that τ is independent of the variables x_{jn} for $j = 1, 3, 5, \dots$. For the principal realization of the basic module of this affine Lie algebra $\hat{sl}_n^{(2)}$, there exists a 'natural' Virasoro algebra. Now assuming that this τ -function also satisfies $L_{-1}\tau = 0$, we show that τ also satisfies the vacuum constraints of the $BW_{1+\infty}$ algebra. The best way to describe $BW_{1+\infty}$ is as a subalgebra of $W_{1+\infty}$. Let ι be a linear anti-involution on $W_{1+\infty}$ defined by:

$$\iota(s) = -s, \quad \iota\left(s\frac{\partial}{\partial s}\right) = -s\frac{\partial}{\partial s} \quad \text{and} \quad \iota(c) = -c, \quad (1.3)$$

then

$$BW_{1+\infty} = \{w \in W_{1+\infty} | \iota(w) = -w\}. \quad (1.4)$$

Let

$$W_{\frac{j}{2}}^{(k+1)} = -s(s^{j+2k}(\frac{\partial}{\partial s^2})^k - (-)^{j+k}(\frac{\partial}{\partial s^2})^k s^{j+2k})s^{-1},$$

we then show that

$$\{W_{\frac{j}{2}}^{(k+1)} + \delta_{j0}c_{k+1}\}\tau = 0 \quad \text{for } j \geq -2k \text{ and } k \leq 0,$$

here c_{k+1} are constants that depend on n .

Many of the results presented in the sections 2–4 are well-known and can be found in e.g. [DJKM], [Sh2] and [Y].

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2. The spin representation of o_∞ , B_∞ and the BKP hierarchy in the fermionic picture

2.1. Let $\overline{gl_\infty}$ be the Lie algebra of complex infinite dimensional matrices such that all nonzero entries are within a finite distance from the main diagonal, i.e.,

$$\overline{gl_\infty} = \{(a_{ij})_{i,j \in \mathbf{Z}} | g_{ij} = 0 \text{ if } |i - j| > 0\}.$$

The elements E_{ij} , the matrix with the (i, j) -th entry 1 and 0 elsewhere, for $i, j \in \mathbf{Z}$ form a basis of a subalgebra $gl_\infty \subset \overline{gl_\infty}$. The Lie algebra gl_∞ has a universal central extension $A_\infty = \overline{gl_\infty} \oplus \mathbf{C}c_A$ with the Lie bracket defined by

$$[a + \alpha c_A, b + \beta c_A] = ab - ba + \mu(a, b)c_A \quad (2.1)$$

for $a, b \in \overline{gl_\infty}$ and $\alpha, \beta \in \mathbf{C}$; here μ is the following 2-cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}(\theta(i) - \theta(j)), \quad (2.2)$$

where the function $\theta : \mathbf{R} \rightarrow \mathbf{R}$ is defined by

$$\theta(i) = \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \leq 0, \end{cases} \quad (2.3)$$

2.2. Define on $\overline{gl_\infty}$ the following linear anti-involution:

$$\iota(E_{jk}) = (-)^{j+k} E_{-k, -j} \quad (2.4)$$

Using this anti-involution we define the Lie algebra $\overline{o_\infty}$ as a subalgebra of $\overline{gl_\infty}$:

$$\overline{o_\infty} = \{a \in \overline{gl_\infty} | \iota(a) = -a\} \quad (2.5)$$

The elements $F_{jk} = E_{-j, k} - (-)^{j+k} E_{-k, j} = -(-)^{j+k} F_{kj}$ with $j < k$ form a basis of $o_\infty = \overline{o_\infty} \cap gl_\infty$. The 2-cocycle μ on gl_∞ induces a 2-cocycle on $\overline{o_\infty}$, and hence we can define a central extension $B_\infty = \overline{o_\infty} \oplus \mathbf{C}c_B$ of $\overline{o_\infty}$, with Lie bracket

$$[a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2}\mu(a, b)c_B \quad (2.6)$$

for $a, b \in \overline{o_\infty}$ and $\alpha, \beta \in \mathbf{C}$.

2.3. We now want to consider highest weight representations of o_∞ and B_∞ . For this purpose we introduce the Clifford algebra BCl as the associative algebra on the generators ϕ_j , $j \in \mathbf{Z}$, called *neutral free fermions*, with defining relations

$$\phi_i \phi_j + \phi_j \phi_i = (-)^i \delta_{i, -j}. \quad (2.7)$$

We define the spin module V over BCl as the irreducible module with highest weight vector the *vacuum vector* $|0\rangle$ satisfying

$$\phi_j |0\rangle = 0 \quad \text{for } j > 0. \quad (2.8)$$

The elements $\phi_{j_1} \phi_{j_2} \cdots \phi_{j_p} |0\rangle$ with $j_1 < j_2 < \cdots < j_p \leq 0$ form a basis of V . Then

$$\begin{aligned} \pi(F_{jk}) &= \frac{(-)^j}{2} (\phi_j \phi_k - \phi_k \phi_j), \\ \hat{\pi}(F_{jk}) &= (-)^j : \phi_j \phi_k :, \\ \hat{\pi}(c_B) &= I, \end{aligned} \quad (2.9)$$

where the normal ordered product $:$ is defined as follows

$$: \phi_j \phi_k := \begin{cases} \phi_j \phi_k & \text{if } k > j, \\ \frac{1}{2}(\phi_j \phi_k - \phi_k \phi_j) & \text{if } j = k, \\ -\phi_k \phi_j & \text{if } k < j, \end{cases} \quad (2.10)$$

define representations of o_∞ , respectively B_∞ .

When restricted to o_∞ and B_∞ , the spin module V breaks into the direct sum of two irreducible modules. To describe this decomposition we define a \mathbf{Z}_2 -gradation on V by introducing a chirality operator χ satisfying $\chi|0\rangle = |0\rangle$, $\chi\phi_j + \phi_j\chi = 0$ for all $j \in \mathbf{Z}$, then

$$V = \bigoplus_{\alpha \in \mathbf{Z}_2} V_\alpha \quad \text{where } V_\alpha = \{v \in V | \chi v = (-)^\alpha v\}.$$

Each module V_α is an irreducible highest weight module with highest weight vector $|0\rangle$, $|1\rangle = \sqrt{2}\phi_0|0\rangle$ for V_0 , V_1 , respectively, in the sense that

$$\begin{aligned} \pi(F_{-i,j})|\alpha\rangle &= \hat{\pi}(F_{-i,j})|\alpha\rangle = 0 \quad \text{for } i < j, \\ \pi(F_{-i,i}) &= -\frac{(-)^i}{2}|\alpha\rangle \quad \text{for } i > 0, \\ \hat{\pi}(F_{-i,i}) &= 0. \end{aligned} \tag{2.11}$$

2.4. Now we define the operator Q on V by

$$\begin{aligned} Q|0\rangle &= \sqrt{2}\phi_0|0\rangle, \\ Q\phi_j &= \phi_j Q \quad \text{for all } j \in \mathbf{Z}. \end{aligned} \tag{2.12}$$

Clearly $Q^2 = I$. Let S be the following operator on $V \otimes V$:

$$S = \sum_{j \in \mathbf{Z}} (-)^j \phi_j \otimes \phi_{-j}. \tag{2.13}$$

Then

$$S(|0\rangle \otimes |0\rangle) = \phi_0|0\rangle \otimes \phi_0|0\rangle = \frac{1}{2}Q|0\rangle \otimes Q|0\rangle.$$

Notice that both Q and S commute with the action of o_∞ . Let $\tau \in V_0$, then we define the BKP equation (in the fermionic picture) to be the following equation:

$$S(\tau \otimes \tau) = \frac{1}{2}Q\tau \otimes Q\tau. \tag{2.14}$$

One can show [H] that there exists a group G for which τ an element of the group orbit of the vacuum vector $|0\rangle$ is, if and only if τ satisfies (2.14). But since we will not use the group in the rest of this paper, we will not prove this statement here.

3. Vertex operators and the BKP hierarchy in the bosonic picture

3.1. Define the following two generating series (fermionic fields):

$$\phi^\pm(z) = \sum_{j \in \mathbf{Z}} \phi_j^\pm z^{-\frac{j}{2} - \frac{1}{2}} = \sum_{j \in \mathbf{Z}} (\pm)^j \phi_j z^{-\frac{j}{2} - \frac{1}{2}}.$$

Using this we define

$$\alpha(z) = \sum_{j \in \frac{1}{2} + \mathbf{Z}} \alpha_j z^{-j-1} = \frac{1}{2} : \phi^+(z) \phi^-(z) :, \quad (3.1)$$

then one has (see e.g. [tKL] for details):

$$\begin{aligned} [\alpha_j, \phi^\pm(z)] &= \pm z^j \phi^\pm(z), \\ [\alpha_j, \alpha_k] &= j \delta_{j, -k} \end{aligned}$$

and

$$\phi^\pm(z) = \frac{Q}{\sqrt{2}} \exp(\mp \sum_{j < 0} \frac{z^{-j}}{j} \alpha_j) \exp(\mp \sum_{j > 0} \frac{z^{-j}}{j} \alpha_j). \quad (3.2)$$

Then it is straightforward that one has the following isomorphism (see [tKL]): $\sigma : V \rightarrow \mathbf{C}[\theta, x_1, x_3, \dots]$, where $\theta^2 = 0$, $x_i x_j = x_j x_i$, $\theta x_j = x_j \theta$ and $V_\alpha = \theta^\alpha \mathbf{C}[x_1, x_3, \dots]$. Now $\sigma(|0\rangle) = 1$ and

$$\begin{aligned} \sigma \alpha_j \sigma^{-1} &= \begin{cases} -j x_{2j} & \text{if } j < 0, \\ \frac{\partial}{\partial x_{2j}} & \text{if } j > 0, \end{cases} \\ \sigma Q \sigma^{-1} &= \theta + \frac{\partial}{\partial \theta}. \end{aligned} \quad (3.3)$$

Hence

$$\sigma \phi^\pm(z) \sigma^{-1} = \frac{\theta + \frac{\partial}{\partial \theta}}{\sqrt{2}} z^{-\frac{1}{2}} \exp(\pm \sum_{j > 0, \text{odd}} x_j z^{\frac{j}{2}}) \exp(\mp 2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial x_j} \frac{z^{-j}}{j}). \quad (3.4)$$

3.2. We first rewrite the BKP hierarchy (2.14):

$$\text{Res}_{z=0} dz \phi^+(z) \tau \otimes \phi^-(z) \tau = \frac{1}{2} Q \tau \otimes Q \tau. \quad (3.5)$$

Here $\text{Res}_{z=0} dz \sum_j f_j z^j = f_{-1}$. Now replace z by z^2 and use (3.4), then (3.5) is equivalent to

$$\text{Res}_{z=0} \frac{dz}{z} \exp \sum_{j > 0, \text{odd}} x_j z^j \exp(-2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial x_j} \frac{z^{-j}}{j}) \tau \otimes \exp - \sum_{j > 0, \text{odd}} x_j z^j \exp(2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial x_j} \frac{z^{-j}}{j}) \tau = \tau \otimes \tau. \quad (3.6)$$

Equation (3.6) is called the BKP hierarchy in the bosonic picture. It is straightforward, using change of variables and Taylor's formula, to rewrite (3.6) into a generating series of Hirota bilinear equations (see e.g. [DJKM]).

4. The BKP hierarchy in terms of formal pseudo-differential operators

4.1. We start by reviewing some of the basic theory of formal pseudo-differential operators (see e.g. [DJKM], [Sh1] and [KL]). We shall work over the algebra A of formal power series over \mathbf{C} in indeterminates $x = (x^k)$, where $k = 1, 3, 5, \dots$. The indeterminate x_1 will be viewed as variables and x_k with $k \geq 3$ as parameters. Let $\partial = \frac{\partial}{\partial x_1}$, a *formal matrix pseudo-differential operator* is an expression of the form

$$P(x, \partial) = \sum_{j \leq N} P_j(x) \partial^j, \quad (4.1)$$

where $P_j \in A$. Let Ψ denote the vector space over \mathbf{C} of all expressions (4.1). We have a linear isomorphism $S : \Psi \rightarrow A((z))$ given by $S(P(x, \partial)) = P(x, z)$. The series $P(x, z)$ in indeterminates x and z is called the *symbol* of $P(x, \partial)$.

Now we may define a product \circ on Ψ making it an associative algebra:

$$S(P \circ Q) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n S(P)}{\partial z^n} \partial^n S(Q). \quad (4.2)$$

From now on, we shall drop the multiplication sign \circ when no ambiguity may arise. One defines the differential part of $P(x, \partial)$ by $P_+(x, \partial) = \sum_{j=0}^N P_j(x) \partial^j$, and let $P_- = P - P_+$. We have the corresponding vector space decomposition:

$$\Psi = \Psi_- \oplus \Psi_+. \quad (4.3)$$

One defines a linear map $*$: $\Psi \rightarrow \Psi$ by the following formula:

$$\left(\sum_j P_j \partial^j \right)^* = \sum_j (-\partial)^j \circ P_j. \quad (4.4)$$

Note that $*$ is an anti-involution of the algebra Ψ . There exists yet another anti-involution, viz. (see also [Sh2])

$$\iota^* P = \partial^{-1} P^* \partial \quad (4.5)$$

Introduce the following notation

$$z \cdot x = \sum_{k=1}^{\infty} x_{2k-1} z^{2k-1}.$$

The algebra Ψ acts on the space U_+ (resp. U_-) of formal oscillating matrix functions of the form

$$\sum_{j \leq N} P_j z^j e^{z \cdot x} \quad (\text{resp. } \sum_{j \leq N} P_j z^j e^{-z \cdot x}), \quad \text{where } P_j \in A,$$

in the obvious way:

$$P(x) \partial^j e^{\pm z \cdot x} = P(x) (\pm z)^j e^{\pm z \cdot x}.$$

One has the following fundamental lemma (see [DJKM], [[K],[KL]] or [Sh1]).

Lemma 4.1. *If $P, Q \in \Psi$ are such that*

$$\text{Res}_{z=0} dz P(x, \partial) e^{z \cdot x} Q(x', \partial') e^{-z \cdot x'} = 0, \quad (4.6)$$

then $(P \circ Q^)_- = 0$.*

4.2. Divide (3.6) by τ , remove the tensor symbol \otimes and write x , respectively x' for the first, respectively the second, term of the tensor product, then (3.6) is equivalent to

$$\text{Res}_{z=0} \frac{dz}{z} w(x, z) w(x', -z) = 1, \quad (4.7)$$

where $w(x, z) = P(x, z)e^{x \cdot z} = \sum_{i \geq 0} P_i z^{-i} e^{x \cdot z}$ and

$$\begin{aligned} P(x, z) &= \frac{\exp(-2 \sum_{j>0} \frac{\partial}{\partial x_j} \frac{z^{-j}}{j}) \tau(x)}{\tau(x)} \\ &= \frac{\tau(x_1 - \frac{2}{z}, x_3 - \frac{2}{3z^3}, \dots)}{\tau(x)} =: \frac{\tilde{\tau}(x, z)}{\tau(x)}. \end{aligned} \quad (4.8)$$

Notice that $P_0 = 1$. Now differentiate (4.7) to x_k , then we obtain

$$\text{Res}_{z=0} \frac{dz}{z} \left(\frac{\partial P(x, z)}{\partial x_k} + P(x, z) z^k \right) e^{x \cdot z} P(x', -z) e^{-x' \cdot z} = 0. \quad (4.9)$$

Now using lemma 4.1 we deduce that

$$\left(\left(\frac{\partial P}{\partial x_k} + P \partial^k \right) \partial^{-1} P^* \right)_- = 0.$$

From the case $k = 1$ we then deduce that $P^* = \partial P^{-1} \partial^{-1}$ or

$$P^{-1} = \iota^*(P), \quad (4.10)$$

if $k \neq 1$, one thus obtains

$$\frac{\partial P}{\partial x_k} = -(P \partial^k P^{-1} \partial^{-1})_- \partial P. \quad (4.11)$$

Since k is odd, $\iota^*(P \partial^k P^{-1}) = -P \partial^k P^{-1}$ and hence $(P \partial^k P^{-1} \partial^{-1})_- \partial = (P \partial^k P^{-1})_-$. So (4.11) turns into Sato's equation:

$$\frac{\partial P}{\partial x_k} = -(P \partial^k P^{-1})_- P. \quad (4.12)$$

4.3. Define the operators

$$L = P \partial P^{-1}, \quad \Gamma = \sum_{j>0} j x_j \partial^{j-1} \quad \text{and} \quad M = P \Gamma P^{-1}. \quad (4.13)$$

Then $[L, M] = 1$ and $\iota^*(L) = -L$. Let $B_k = (L^k)_+$, using (4.12) one deduces the following Lax equations:

$$\begin{aligned} \frac{\partial L}{\partial x_k} &= [B_k, L], \\ \frac{\partial M}{\partial x_k} &= [B_k, M]. \end{aligned} \quad (4.14)$$

The first equation of (4.14) is equivalent to the following Zakharov Shabat equation:

$$\frac{\partial B_j}{\partial x_k} - \frac{\partial B_k}{\partial x_j} = [B_k, B_j], \quad (4.15)$$

which are the compatibility conditions of the following linear problem for $w = w(x, z)$:

$$Lw = zw, \quad Mw = \frac{\partial w}{\partial z} \quad \text{and} \quad \frac{\partial w}{\partial x_k} = B_k w. \quad (4.16)$$

4.4. The formal adjoint of the wave function w is (see [DJKM]):

$$\begin{aligned} w^* &= w^*(x, z) = P^{*-1} e^{-x \cdot z} \\ &= \partial P \partial^{-1} e^{-x \cdot z}. \end{aligned} \quad (4.17)$$

Now $L^* = -\partial L \partial^{-1} = -\partial P \partial P^{-1} \partial^{-1}$ and $M^* = \partial P \partial^{-1} \Gamma \partial P^{-1} \partial^{-1}$, so $[L^*, M^*] = -1$ and

$$L^* w^* = z w^*, \quad M^* w^* = -\frac{\partial w^*}{\partial z} \quad \text{and} \quad \frac{\partial w^*}{\partial x_k} = -(L^*)_+ w^* = -B_k^* w^*. \quad (4.18)$$

Finally, notice that by differentiating the bilinear identity (4.7) to x'_1 we obtain

$$\text{Res}_{z=0} dz w(x, z) w^*(x', z) = 0. \quad (4.19)$$

5. The n -th reduced BKP hierarchy

5.1. From now on we assume that n is an odd integer. Let $\omega = e^{\frac{2\pi i}{n}}$, then it is well-known [DJKM], [tKL] that the fields

$$A_j(z) =: \phi^+(z)\phi^-(\omega^{2j}z) : \quad \text{for } j = 1, 2, \dots, n \quad (5.1)$$

generate the principal realization of the basic representation of the Lie algebra $\hat{gl}_n^{(2)}$. Using (3.4), one can express the fields (5.1) for $j \neq n$ in terms of the x_k and $\frac{\partial}{\partial x_k}$'s. These fields for $j \neq n$ are independent of x_{kn} and $\frac{\partial}{\partial x_{kn}}$. Hence in order to describe the representation theory of $\hat{sl}_n^{(2)}$ one only has to remove x_{kn} and $\frac{\partial}{\partial x_{kn}}$ in $A_n(z) = 2\alpha(z)$.

5.2. The reduction of the BKP hierarchy to $\hat{sl}_n^{(2)}$, considered in the previous subsection, is called the n -th reduced BKP hierarchy. Hence, from now on we will call a BKP τ -function n -reduced, if it satisfies

$$\frac{\partial \tau}{\partial x_{kn}} = 0 \quad \text{for } k = 1, 3, 5, \dots \quad (5.2)$$

Using Sato's equation (4.12) this implies the following two equivalent conditions:

$$\begin{aligned} \frac{\partial w}{\partial x_{kn}} &= z^{kn} w, \\ (L^{kn})_- &= 0 \quad \text{for } k = 1, 3, 5, \dots \end{aligned}$$

Hence L^n is a differential operator.

6. The string equation

6.1. The principal realization of $\hat{gl}_n^{(2)}$ has a natural Virasoro algebra. In [tKL] it was shown that the following two sets of operators have the same action on V ($k \in \mathbf{Z}$):

$$\begin{aligned} L_k &= \frac{1}{2n} \sum_{j \in \frac{1}{2} + \mathbf{Z}} : \alpha_{-j} \alpha_{j+nk} : + \delta_{k,0} \left(\frac{1}{16n} + \frac{n^2 - 1}{24n} \right), \\ H_k &= \sum_{j \in \mathbf{Z}} \frac{j}{4n} : \phi_{-j}^+ \phi_{j+2kn}^- : + \delta_{k,0} \left(\frac{1}{16n} + \frac{n^2 - 1}{24n} \right). \end{aligned} \quad (6.1)$$

So $L_k = H_k$ and

$$\begin{aligned} [L_k, \phi_j^\pm] &= -\left(\frac{j}{2n} + \frac{k}{2}\right) \phi_{j+2kn}^\pm, \\ [L_k, L_j] &= (k - j) L_{k+j} + \delta_{k,-j} \frac{k^3 - k}{12} n. \end{aligned}$$

Using (3.3), we can rewrite L_{-1} in terms of the x_k and $\frac{\partial}{\partial x_k}$'s:

$$L_{-1} = \frac{1}{8n} \sum_{k=1, \text{odd}}^{2n-1} k(2n-k) x_k x_{2n-k} + \frac{1}{2n} \sum_{k=1, \text{odd}}^{\infty} (k+2n) x_{k+2n} \frac{\partial}{\partial x_k}. \quad (6.2)$$

We now define in analogy with the untwisted \hat{sl}_2 case, i.e. the KdV hierarchy, the *string equation* to be the following restriction on $\tau \in V_0$:

$$L_{-1} \tau = 0. \quad (6.3)$$

From this we deduce (see also [D], [L]) that also

$$\begin{aligned} L_{-1} \tilde{\tau}(x, z) &= \left\{ \frac{1}{8n} \sum_{k=1, \text{odd}}^{2n-1} k(2n-k) \left(x_k - \frac{2}{kz^k} \right) \left(x_{2n-k} - \frac{2}{(2n-k)z^{2n-k}} \right) \right. \\ &\quad \left. + \frac{1}{2n} \sum_{k=1, \text{odd}}^{\infty} (k+2n) \left(x_{k+2n} - \frac{2}{(k+2n)z^{k+2n}} \right) \frac{\partial}{\partial x_k} \right\} \tilde{\tau}(x, z) = 0. \end{aligned}$$

Now calculating

$$-\frac{\tilde{\tau}(x, z)L_{-1}\tau(x)}{\tau(x)^2} + \frac{L_{-1}\tilde{\tau}(x, z)}{\tau(x)}$$

explicitly, we deduce that

$$\begin{aligned} & \frac{1}{2n} \sum_{k=1, \text{odd}}^{\infty} (k+2n)x_{k+2n} \frac{\partial(\tau(x)^{-1}\tilde{\tau}(x, z))}{\partial x_k} + \frac{1}{2} z^{-2n} \frac{\tilde{\tau}(x, z)}{\tau(x)} - \frac{1}{2n} \sum_{k=1, \text{odd}}^{2n-1} \frac{kx_k}{z^{2n-k}} \frac{\tilde{\tau}(x, z)}{\tau(x)} \\ & - \frac{1}{n} \frac{1}{\tau(x)} \sum_{k=1, \text{odd}}^{\infty} \frac{1}{z^{2n+k}} \frac{\partial \tilde{\tau}(x, z)}{\partial x_k} = 0. \end{aligned}$$

Now compare this with the symbol of $(\frac{1}{2n}ML^{1-2n})_-P$, which is

$$\begin{aligned} S((\frac{1}{2n}ML^{1-2n})_-P) &= -\frac{1}{2n} \sum_{k=1, \text{odd}}^{\infty} (k+2n)x_{k+2n} \frac{\partial(\tau(x)^{-1}\tilde{\tau}(x, z))}{\partial x_k} + \frac{1}{2n} \sum_{k=1, \text{odd}}^{2n-1} \frac{kx_k}{z^{2n-k}} \frac{\tilde{\tau}(x, z)}{\tau(x)} \\ &+ \frac{1}{n} \frac{1}{\tau(x)} \sum_{k=1, \text{odd}}^{\infty} \frac{1}{z^{2n+k}} \frac{\partial \tilde{\tau}(x, z)}{\partial x_k}. \end{aligned}$$

We thus conclude that the string equation leads to

$$(\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n})_-P = 0$$

and hence to

$$(\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n})_- = 0. \quad (6.4)$$

So $\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n}$ is a differential operator that , moreover, satisfies

$$[L^{2n}, \frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n}] = 1. \quad (6.5)$$

6.2. Notice that since $(L^n)_- = 0$ one has

$$(\frac{1}{n}ML^{1-n})_- = ((\frac{1}{n}ML^{1-n})_-L^n)_- = L^{-n},$$

so also $\frac{1}{n}ML^{1-n} - L^{-n}$ is a differential operator that satisfies

$$[L^n, \frac{1}{n}ML^{1-n} - L^{-n}] = 1.$$

7. Extra constraints

7.1. From now on we assume that τ is any solution of the BKP hierarchy that satisfies:

$$\begin{aligned} \frac{\partial \tau}{\partial x_{kn}} &= 0 \quad \text{for } k = 1, 3, 5, \dots \quad \text{and} \\ L_{-1} \tau &= 0. \end{aligned}$$

Hence $(L^n)_- = 0$ and $(\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n})_- = 0$. Taking the formal adjoint of these operators one deduces $(\partial L^n \partial^{-1})_- = 0$ and $(\frac{1}{2n}\partial ML^{1-2n}\partial^{-1} - \frac{1}{2}\partial L^{-2n}\partial^{-1})_- = 0$. Hence more generally we have for all $p, q \in \mathbf{Z}_+$:

$$\begin{aligned} ((\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n})^q L^{pn})_- &= 0, \\ (\partial(\frac{1}{2n}ML^{1-2n} - \frac{1}{2}L^{-2n})^q L^{pn} \partial^{-1})_- &= 0. \end{aligned} \tag{7.1}$$

Now using (4.16) and (4.18) one shows the following

Lemma 7.1. *For all $p, q \in \mathbf{Z}_+$ one has*

$$\begin{aligned} \text{Res}_{z=0} dz z^{qn} (\frac{1}{2n} z^{1-n} \frac{\partial}{\partial z} z^{-n})^p (w(x, z)) w^*(x', z) &= 0 \\ \text{Res}_{z=0} dz z^{qn} (\frac{1}{2n} z^{1-n} \frac{\partial}{\partial z} z^{-n})^p (w^*(x, -z)) w(x', -z) &= 0 \end{aligned} \tag{7.2}$$

Proof. The proof of this lemma is similar to the proof of lemma 6.1 of [L] □

In terms of the fermionic fields this means

Corollary 7.2. *For all $p, q \in \mathbf{Z}_+$ one has*

$$\begin{aligned} \text{Res}_{z=0} dz z^{\frac{qn}{2}} (\frac{1}{n} z^{\frac{1-n}{2}} \frac{\partial}{\partial z} z^{\frac{1-n}{2}})^p (\frac{\phi^+(z)\tau}{\tau}) \otimes \partial(\frac{\phi^-(z)\tau}{\tau}) &= 0 \\ \text{Res}_{z=0} dz z^{\frac{qn}{2}} (\frac{1}{n} z^{\frac{1-n}{2}} \frac{\partial}{\partial z} z^{\frac{1-n}{2}})^p (\partial(\frac{\phi^+(z)\tau}{\tau})) \otimes \frac{\phi^-(z)\tau}{\tau} &= 0 \end{aligned} \tag{7.3}$$

7.2. In the rest of this section the following lemma will be crucial:

Lemma 7.3.

$$\phi^+(u)\tau \otimes \frac{\partial}{\partial x_1} (\frac{\phi^-(v)\tau}{\tau}) - \phi^-(v)\tau \otimes \frac{\partial}{\partial x_1} (\frac{\phi^+(u)\tau}{\tau}) = -\text{Res}_{z=0} dz \phi^+(z) : \phi^+(u) \phi^-(v) : \tau \otimes \frac{\partial}{\partial x_1} (\frac{\phi^-(z)\tau}{\tau}).$$

Proof. The bilinear identity (4.19) is equivalent to

$$\text{Res}_{z=0} dz \phi^+(z) \tau \otimes \frac{\partial}{\partial x_1} (\frac{\phi^-(z)\tau}{\tau}) = 0. \tag{7.4}$$

Now let $2(uv)^{\frac{1}{2}} \phi^+(u) \phi^-(v) \otimes 1$ act on this identity, then one obtains:

$$\begin{aligned} \text{Res}_{z=0} \frac{dz}{z} \frac{1 + (v/u)^{\frac{1}{2}}}{1 - (v/u)^{\frac{1}{2}}} \frac{1 - (z/u)^{\frac{1}{2}}}{1 + (z/u)^{\frac{1}{2}}} \frac{1 + (z/v)^{\frac{1}{2}}}{1 - (z/v)^{\frac{1}{2}}} \exp(-\sum_{k<0} \frac{u^{-k} + z^{-k} - v^{-k}}{k} \alpha_k) \times \\ \exp(-\sum_{k>0} \frac{u^{-k} + z^{-k} - v^{-k}}{k} \alpha_k) \tau \otimes \frac{\partial}{\partial x_1} (\frac{\exp(\sum_{k<0} \frac{z^{-k}}{k} \alpha_k) \exp(\sum_{k>0} \frac{z^{-k}}{k} \alpha_k) \tau}{\tau}) = 0. \end{aligned} \tag{7.5}$$

Now using the fact that $\frac{1+w}{1-w} = 2\delta(w) - \frac{1+w^{-1}}{1-w^{-1}}$, then (7.5) reduces to

$$\begin{aligned} & 2(uv)^{\frac{1}{2}}(\phi^+(u)\tau \otimes \frac{\partial}{\partial x_1}(\frac{\phi^-(v)\tau}{\tau}) - \phi^-(v)\tau \otimes \frac{\partial}{\partial x_1}(\frac{\phi^+(u)\tau}{\tau})) \\ & + \text{Res}_{z=0} \frac{dz}{z} \frac{1+(v/u)^{\frac{1}{2}}}{1-(v/u)^{\frac{1}{2}}} \frac{1-(u/z)^{\frac{1}{2}}}{1+(u/z)^{\frac{1}{2}}} \frac{1+(v/z)^{\frac{1}{2}}}{1-(v/z)^{\frac{1}{2}}} \exp(-\sum_{k<0} \frac{u^{-k}+z^{-k}-v^{-k}}{k} \alpha_k) \times \\ & \exp(-\sum_{k>0} \frac{u^{-k}+z^{-k}-v^{-k}}{k} \alpha_k) \tau \otimes \frac{\partial}{\partial x_1}(\frac{\exp(\sum_{k<0} \frac{z^{-k}}{k} \alpha_k) \exp(\sum_{k>0} \frac{z^{-k}}{k} \alpha_k) \tau}{\tau}) = 0. \end{aligned} \quad (7.6)$$

Now the last term on the left-hand-side is equal to

$$\begin{aligned} & \text{Res}_{z=0} dz 2(uv)^{\frac{1}{2}} \phi^+(z) \phi^+(u) \phi^-(v) \tau \otimes \frac{\partial}{\partial x_1}(\frac{\phi^-(z)\tau}{\tau}) = \\ & \text{Res}_{z=0} dz 2(uv)^{\frac{1}{2}} \phi^+(z) : \phi^+(u) \phi^-(v) : \tau \otimes \frac{\partial}{\partial x_1}(\frac{\phi^-(z)\tau}{\tau}). \end{aligned}$$

□

7.3. Define

$$W_{\frac{q}{2}-p}^{(p+1)} = \text{Res}_{z=0} dz z^{\frac{qn}{2}} (\frac{1}{n} y^{\frac{1-n}{2}} \frac{\partial}{\partial y} y^{\frac{1-n}{2}})^p : \phi^+(y) \phi^-(z) : |_{y=z}, \quad (7.7)$$

then from lemma 7.3 and corollary 7.2 we deduce that

$$\text{Res}_{z=0} dz \phi^+(z) W_{\frac{q}{2}-p}^{(p+1)} \tau \otimes \frac{\partial}{\partial x_1}(\frac{\phi^-(z)\tau}{\tau}) = 0,$$

or explicitly in terms of the x_k and $\frac{\partial}{\partial x_k}$'s:

$$\text{Res}_{z=0} \frac{dz}{z} e^{x \cdot z} \exp(-2 \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k}) (W_{\frac{q}{2}-p}^{(p+1)} \tau(x)) \frac{\partial}{\partial x_1'} (e^{-x' \cdot z} \frac{\exp(2 \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k'} \tau(x'))}{\tau(x')}) = 0. \quad (7.8)$$

First take $x_k = x'_k$ for all $k = 1, 3, \dots$, then one deduces that

$$\frac{\partial}{\partial x_1} (\frac{W_{\frac{q}{2}-p}^{(p+1)} \tau(x)}{\tau(x)}) = 0. \quad (7.9)$$

Now divide (7.8) by $\tau(x)$, then

$$\text{Res}_{z=0} dz w(x, z) \exp(-2 \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k}) (\frac{W_{\frac{q}{2}-p}^{(p+1)} \tau(x)}{\tau(x)}) w^*(x', z) = 0. \quad (7.10)$$

Now subtract a multiple of the bilinear identity (4.19), then one obtains

$$\text{Res}_{z=0} dz w(x, z) (\exp(-2 \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k}) - 1) (\frac{W_{\frac{q}{2}-p}^{(p+1)} \tau(x)}{\tau(x)}) w^*(x', z) = 0. \quad (7.11)$$

Define

$$S_{pq} = S_{pq}(x, z) = (\exp(-2 \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k}) - 1) (\frac{W_{\frac{q}{2}-p}^{(p+1)} \tau(x)}{\tau(x)}),$$

then (7.9) implies that $\partial \circ S_{pq}(x, \partial) = S_{pq}(x, \partial) \circ \partial$. Using this and lemma 4.1, we deduce that

$$(PS_{pq}(\partial P \partial^{-1})^*)_- = (PS_{pq} P^{-1})_- = PS_{pq} P^{-1} = 0.$$

So $S_{pq}(x, z) = 0$ and hence

$$\frac{W_{\frac{q}{2}-p}^{(p+1)} \tau(x)}{\tau(x)} = \text{constant} \quad \text{for } p, q \in \mathbf{Z}_+. \quad (7.12)$$

In the next section we will see that the $W_{\frac{q}{2}-p}^{(p+1)}$ form a subalgebra of $W_{1+\infty}$.

8. The $BW_{1+\infty}$ constraints

8.1. The Lie algebras gl_∞ , o_∞ , $\overline{gl_\infty}$ and $\overline{o_\infty}$ all have a natural action on the space of column vectors, viz., let $\mathbf{C}^\infty = \bigoplus_{k \in \mathbf{Z}} e_k$, then $E_{ij}e_k = \delta_{jk}e_i$. By identifying e_k with s^{-k} , we can embed the algebra \mathbf{D} of differential operators on the circle, with basis $-s^{j+k}(\frac{\partial}{\partial s})^k$ ($j \in \mathbf{Z}, k \in \mathbf{Z}_+$), in $\overline{gl_\infty}$:

$$\begin{aligned} \rho : \mathbf{D} &\rightarrow \overline{gl_\infty} \\ \rho(-s^{j+k}(\frac{\partial}{\partial s})^k) &= \sum_{m \in \mathbf{Z}} -m(m-1) \cdots (m-k+1) E_{-m-j, -m} \end{aligned} \quad (8.1)$$

It is straightforward to check that the 2-cocycle μ on $\overline{gl_\infty}$ induces the following 2-cocycle on \mathbf{D} :

$$\mu(-s^{i+j}(\frac{\partial}{\partial s})^j, -s^{k+\ell}(\frac{\partial}{\partial s})^\ell) = \delta_{i,-k}(-)^j j! \ell! \binom{i+j}{j+\ell+1}. \quad (8.2)$$

This cocycle was discovered by Kac and Peterson in [KP] (see also [R], [KR]). In this way we have defined a central extension of \mathbf{D} , which we denote by $W_{1+\infty} = \mathbf{D} \oplus \mathbf{C}c_A$, the Lie bracket on $W_{1+\infty}$ is given by

$$\begin{aligned} [-s^{i+j}(\frac{\partial}{\partial s})^j + \alpha c_A, -s^{k+\ell}(\frac{\partial}{\partial s})^\ell + \beta c_A] = \\ \sum_{m=0}^{\max(j,\ell)} m! \left(\binom{i+j}{m} \binom{\ell}{m} - \binom{k+\ell}{m} \binom{j}{m} \right) (-s^{i+j+k+\ell-m}(\frac{\partial}{\partial s})^{j+\ell-m} + \delta_{i,-k}(-)^j j! \ell! \binom{i+j}{j+\ell+1}) c_A. \end{aligned} \quad (8.3)$$

Let $D_m = s^m \frac{\partial}{\partial s^m}$ and set $D = D_1$, then we can rewrite the elements $-s^{i+j}(\frac{\partial}{\partial s})^j$, viz. ,

$$-s^{i+j}(\frac{\partial}{\partial s})^j = -s^i D(D-1)(D-2) \cdots (D-j+1). \quad (8.4)$$

Then for $k \geq 0$ the 2-cocycle is as follows [KR]:

$$\mu(s^k f(D), s^\ell g(D)) = \begin{cases} \sum_{-k \leq j \leq -1} f(j)g(j+k) & \text{if } k = -\ell \\ 0 & \text{otherwise} \end{cases} \quad (8.5)$$

8.2. Now replace s by $t^{\frac{1}{2}}$ and write $2t^{\frac{1}{2}} \frac{\partial}{\partial t}$ instead of $\frac{\partial}{\partial t^{\frac{1}{2}}}$. Then a new basis of \mathbf{D} is given by $-t^{\frac{j}{2}+k}(\frac{\partial}{\partial t})^k$. It is then straightforward to check that the anti-involution ι defined by (2.4) induces

$$\iota(t^{\frac{1}{2}}) = -t^{\frac{1}{2}}, \quad \iota(t^{\frac{1}{2}} \frac{\partial}{\partial t} t^{-\frac{1}{2}}) = -t^{\frac{1}{2}} \frac{\partial}{\partial t} t^{-\frac{1}{2}} \quad \text{and} \quad \iota(D) = -D. \quad (8.6)$$

Hence, it induces the following anti-involution on \mathbf{D} :

$$\iota(t^{\frac{1}{2}} t^{\frac{j}{2}+k} (\frac{\partial}{\partial t})^k t^{-\frac{1}{2}}) = (-)^{j+k} t^{\frac{1}{2}} (\frac{\partial}{\partial t})^k t^{\frac{j}{2}+k} t^{-\frac{1}{2}}. \quad (8.7)$$

Define $\mathbf{D}^B = \mathbf{D} \cap \overline{o_\infty} = \{w \in \mathbf{D} | \iota(w) = -w\}$, it is spanned by the elements

$$\begin{aligned} w_{\frac{j}{2}}^{(k+1)} &= -t^{\frac{1}{2}} (t^{\frac{j}{2}+k} (\frac{\partial}{\partial t})^k - (-)^{j+k} (\frac{\partial}{\partial t})^k t^{\frac{j}{2}+k}) t^{-\frac{1}{2}} \\ &= -t^{\frac{1}{2}} (t^{\frac{j}{2}+k} (\frac{\partial}{\partial t})^k - (-)^{j+k} \sum_{\ell=0}^k \binom{k}{\ell} \ell! \binom{\frac{j}{2}+k}{\ell} t^{\frac{j}{2}+k-\ell} (\frac{\partial}{\partial t})^{k-\ell}) t^{-\frac{1}{2}}. \end{aligned} \quad (8.8)$$

The restriction of the 2-cocycle μ on \mathbf{D} , given by (8.2), induces a 2-cocycle on \mathbf{D}^B , which we shall not calculate explicitly here. It defines a central extension $BW_{1+\infty} = \mathbf{D}^B \oplus \mathbf{C}c_B$ of \mathbf{D}^B , with Lie bracket

$$[a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2} \mu(a, b) c_B,$$

for $a, b \in \mathbf{D}^B$ and $\alpha, \beta \in \mathbf{C}$.

8.3. We work out

$$\begin{aligned}
: \frac{\partial^p \phi^+(z)}{\partial z^p} \phi^-(z) : &= \sum_{k, \ell \in \mathbf{Z}} -p! \binom{\ell - \frac{1}{2}}{p} \hat{\pi}(F_{k+\ell, -\ell}) z^{-\frac{k}{2} - p - 1} \\
&= \sum_{k, \ell \in \mathbf{Z}} -p! \binom{\ell - \frac{1}{2}}{p} (\hat{\pi}(E_{-k-\ell, -\ell}) - (-)^k \hat{\pi}(E_{\ell, k+\ell})) z^{-\frac{k}{2} - p - 1} \\
&= \sum_{k, \ell \in \mathbf{Z}} -p! \left(\binom{\ell - \frac{1}{2}}{p} - (-)^k \binom{-k - \ell - \frac{1}{2}}{p} \right) \hat{\pi}(E_{-k-\ell, -\ell}) z^{-\frac{k}{2} - p - 1} \\
&= \sum_{k, \ell \in \mathbf{Z}} -p! \left(\binom{\ell - \frac{1}{2}}{p} - (-)^{k+p} \binom{k + \ell + p - \frac{1}{2}}{p} \right) \hat{\pi}(E_{-k-\ell, -\ell}) z^{-\frac{k}{2} - p - 1} \\
&= \hat{\pi}(-t^{\frac{1}{2}} (t^{\frac{k}{2} + p} (\frac{\partial}{\partial t})^p - (-)^{k+p} (\frac{\partial}{\partial t})^p t^{\frac{k}{2} + p}) t^{-\frac{1}{2}}) z^{-\frac{k}{2} - p - 1} \\
&= \hat{\pi}(w_{\frac{k}{2}}^{(p+1)}) z^{-\frac{k}{2} - p - 1}.
\end{aligned} \tag{8.9}$$

8.4. We want to calculate $W_{\frac{q}{2}-p}^{(p+1)}$. For this purpose we write

$$(z^{\frac{1-n}{2}} \frac{\partial}{\partial z} z^{\frac{1-n}{2}})^p = \sum_{\ell=0}^p c(\ell, p) z^{-np+\ell} (\frac{\partial}{\partial z})^\ell.$$

Then

$$\begin{aligned}
W_{\frac{q}{2}-p}^{(p+1)} &= \text{Res}_{z=0} dz z^{\frac{qn}{2}} \left(\frac{1}{n} y^{\frac{1-n}{2}} \frac{\partial}{\partial y} y^{\frac{1-n}{2}} \right)^p : \phi^+(y) \phi^-(z) : |_{y=z} \\
&= \text{Res}_{z=0} dz \frac{1}{n^p} \sum_{\ell=0}^p c(\ell, p) z^{\frac{qn}{2} - np + \ell} : \frac{\partial^\ell \phi^+(z)}{\partial z^\ell} \phi^-(z) : \\
&= \frac{1}{n^p} \sum_{\ell=0}^p c(\ell, p) \hat{\pi}(w_{\frac{qn}{2} - pn}^{(\ell+1)}) \\
&= \hat{\pi} \left(\frac{1}{n^p} \sum_{\ell=0}^p c(\ell, p) (-t^{\frac{1}{2}} (t^{\frac{qn}{2} - np + \ell} (\frac{\partial}{\partial t})^\ell) t^{-\frac{1}{2}} - \iota(-t^{\frac{1}{2}} (t^{\frac{qn}{2} - np + \ell} (\frac{\partial}{\partial t})^\ell) t^{-\frac{1}{2}})) \right) \\
&= \hat{\pi} \left(\frac{1}{n^p} (-t^{\frac{1}{2}} (t^{\frac{qn}{2}} (\frac{\partial}{\partial t} t^{\frac{1-n}{2}})^p) t^{-\frac{1}{2}}) - \iota \left(\frac{1}{n^p} (-t^{\frac{1}{2}} (t^{\frac{qn}{2}} (t^{\frac{1-n}{2}} \frac{\partial}{\partial t} t^{\frac{1-n}{2}})^p) t^{-\frac{1}{2}}) \right) \right) \\
&= \sum_{k \in \mathbf{Z}} \hat{\pi} \left(-t^{\frac{1}{2}} \left(\frac{1}{n^p} (t^{\frac{qn}{2}} (\frac{\partial}{\partial t} t^{\frac{1-n}{2}})^p - (-)^{p+q} \frac{1}{n^p} (t^{\frac{1-n}{2}} \frac{\partial}{\partial t} t^{\frac{1-n}{2}})^p t^{\frac{qn}{2}}) t^{-\frac{1}{2}} \right) \right) \\
&= \sum_{k \in \mathbf{Z}} \hat{\pi} \left(-\lambda^{\frac{1}{2}} (\lambda^{\frac{q}{2}} (\frac{\partial}{\partial \lambda})^p - (-)^{p+q} (\frac{\partial}{\partial \lambda})^p \lambda^{\frac{q}{2}}) \lambda^{-\frac{1}{2}} \right),
\end{aligned} \tag{8.10}$$

where $\lambda = t^n = s^{2n}$. Hence, from this it is obvious that the elements $W_{\frac{k}{2}}^{(p+1)}$, together with c_B span a $BW_{1+\infty}$ -algebra with $c_B = nI$.

9. The calculation of the constants

9.1. In order to determine the constants on the right-hand-side of (7.12), we notice that

$$\begin{aligned} 0 &= [W_{-1}^{(2)}, -\frac{1}{q+2}W_{\frac{q}{2}-p+1}^{(p+1)}]\tau \\ &= (W_{\frac{q}{2}-p}^{(p+1)} + \frac{1}{2}\mu(W_{-1}^{(2)}, -\frac{1}{q+2}W_{\frac{q}{2}-p+1}^{(p+1)}))\tau. \end{aligned} \quad (9.1)$$

It is clear that the cocycle term of (9.1) is 0, except when $q = 2p$.

Now

$$W_{-1}^{(2)} = -2\lambda^{\frac{1}{2}}\frac{\partial}{\partial\lambda}\lambda^{-\frac{1}{2}} = s^{-2n} - \frac{1}{n}s^{1-2n}\frac{\partial}{\partial s} = -\frac{s^{-2n}}{n}(D-n)$$

and

$$\begin{aligned} & -\frac{1}{2p+2}W_1^{(p+1)} \\ &= \frac{1}{2p+2}[s^{3n}D_{2n}(D_{2n}-1)\cdots(D_{2n}-p+1)s^{-n} - \iota(s^{3n}D_{2n}(D_{2n}-1)\cdots(D_{2n}-p+1)s^{-n})] \\ &= \frac{1}{2p+2}[s^{2n}(D_{2n}-\frac{1}{2})(D_{2n}-\frac{3}{2})\cdots(D_{2n}-p+\frac{1}{2}) - \iota(s^{2n}(D_{2n}-\frac{1}{2})(D_{2n}-\frac{3}{2})\cdots(D_{2n}-p+\frac{1}{2}))] \\ &= \frac{1}{2p+2}\frac{s^{2n}}{(2n)^p}[(D-n)(D-3n)\cdots(D-(2p-1)n) - (-)^p(D+(2p+1)n)(D+(2p-1)n)\cdots(D+3n)]. \end{aligned}$$

Then using (8.5) we deduce that

$$\begin{aligned} & \mu(W_{-1}^{(2)}, -\frac{1}{2p+2}W_1^{(p+1)}) \\ &= \frac{1}{p+1}\left(\frac{1}{2n}\right)^{p+1} \sum_{-2n \leq j \leq -1} [(j+n)(j-n)\cdots(j-(2p-1)n) - (-)^p(j+(2p+1)n)(j+(2p-1)n)\cdots(j+n)] \\ &= \frac{1}{p+1}\left(\sum_{-\frac{1}{2} \leq k \in \frac{1}{2n}\mathbf{Z} < \frac{1}{2}} + \sum_{-\frac{1}{2} < k \in \frac{1}{2n}\mathbf{Z} \leq \frac{1}{2}}\right) k(k-1)(k-2)\cdots(k-p). \end{aligned} \quad (9.2)$$

Hence we can state the main Theorem of this paper:

Theorem 9.1. *The following two constraints on a BKP τ -function are equivalent:*

$$(1) : \quad \frac{\partial\tau}{\partial x_{jn}} = 0 \quad \text{for } j = 1, 3, 5, \dots \quad \text{and}$$

$$\left\{ \sum_{k=1, \text{odd}}^{2n-1} k(2n-k)x_k x_{2n-k} + 4 \sum_{k=1, \text{odd}}^{\infty} (2n+k)x_{2n+k} \frac{\partial}{\partial x_k} \right\} \tau = 0.$$

$$(2) : \quad \left\{ W_{\frac{q}{2}-p}^{(p+1)} + \frac{\delta_{2q,p}}{2} c_{p+1} \right\} \tau = 0,$$

for $p, q \in \mathbf{Z}_+$, where

$$c_{p+1} = \frac{1}{p+1} \left(\sum_{-\frac{1}{2} \leq k \in \frac{1}{2n}\mathbf{Z} < \frac{1}{2}} + \sum_{-\frac{1}{2} < k \in \frac{1}{2n}\mathbf{Z} \leq \frac{1}{2}} \right) k(k-1)(k-2)\cdots(k-p).$$

Proof. The proof of this theorem is now obvious, since (2) clearly implies (1) □

Notice that $c_1 = 0$, $c_2 = \frac{2n^2+1}{6n} = 8(\frac{1}{16n} + \frac{n^2-1}{24n})$.

10. Appendix

In this appendix we show that it is possible to express

$$\hat{W}_{\frac{q}{2}-p}^{(p+1)} = W_{\frac{q}{2}-p}^{(p+1)} + \frac{\delta_{2q,p}}{2} c_{p+1}$$

in terms of the α_j 's. Recall (7.7):

$$\begin{aligned} W_{\frac{q}{2}-p}^{(p+1)} &= \text{Res}_{z=0} dz z^{\frac{qn}{2}} \left(\frac{1}{n} y^{\frac{1-n}{2}} \frac{\partial}{\partial y} y^{\frac{1-n}{2}} \right)^p : \phi^+(y) \phi^-(z) : |_{y=z} \\ &= \text{Res}_{z=0} dz z^{\frac{(q+1)n-1}{2}} \left(\frac{\partial}{\partial y^n} \right)^p y^{\frac{1-n}{2}} : \phi^+(y) \phi^-(z) : |_{y=z} \end{aligned} \quad (10.1)$$

Using (3.2), it is straightforward to check that

$$(y^n - z^n) y^{\frac{1-n}{2}} : \phi^+(y) \phi^-(z) : = \frac{1}{2} \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) y^{\frac{n-k}{2}} z^{\frac{k-1}{2}} (X(y, z) - 1), \quad (10.2)$$

where

$$X(y, z) = \exp\left(-\sum_{k<0} \frac{y^{-k} - z^{-k}}{k} \alpha_k\right) \exp\left(-\sum_{k>0} \frac{y^{-k} - z^{-k}}{k} \alpha_k\right). \quad (10.3)$$

Hence

$$\begin{aligned} W_{\frac{q}{2}-p}^{(p+1)} &= \text{Res}_{z=0} \frac{dz}{z} \frac{1}{2p+2} \left(\frac{\partial}{\partial y^n} \right)^{p+1} \left\{ \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) (y^n)^{\frac{n-k}{2n}} (z^n)^{\frac{(q+1)n+k}{2n}} (X(y, z) - 1) |_{y=z} \right\} \\ &= \text{Res}_{z=0} \frac{dz}{z} \frac{1}{2p+2} \sum_{\ell=0}^p \binom{p+1}{\ell} \left(\frac{\partial}{\partial y} \right)^\ell \left\{ \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) y^{\frac{n-k}{2n}} z^{\frac{(q+1)n+k}{2n}} \frac{\partial^{p+1-\ell} X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^{p+1-\ell}} \right\} |_{y=z} \\ &= \text{Res}_{z=0} \frac{dz}{z} \frac{1}{2p+2} \sum_{\ell=0}^p \binom{p+1}{\ell} \ell! \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) \binom{\frac{n-k}{2n}}{\ell} z^{\frac{q}{2}+1-\ell} \frac{\partial^{p+1-\ell} X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^{p+1-\ell}} |_{y=z}. \end{aligned} \quad (10.4)$$

Notice that $W_{\frac{q}{2}}^{(p+1)} = w_{\frac{q}{2}}^{(p+1)}$ for $n = 1$.

Since

$$c_\ell = (\ell-1)! \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) \binom{\frac{n-k}{2n}}{\ell},$$

one finds that

$$\begin{aligned} \hat{W}_{\frac{q}{2}-p}^{(p+1)} &= \text{Res}_{z=0} \frac{dz}{z} \frac{1}{2p+2} \left(\frac{\partial}{\partial y^n} \right)^{p+1} \left\{ \left(\sum_{k=0}^{2n-1} + \sum_{k=1}^{2n} \right) (y^n)^{\frac{n-k}{2n}} (z^n)^{\frac{(q+1)n+k}{2n}} X(y, z) |_{y=z} \right\} \\ &= \text{Res}_{z=0} dz \frac{z^{\frac{q}{2}}}{2p+2} \left(4n \frac{\partial^{p+1} X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^{p+1}} + \sum_{\ell=2}^{p+1} \binom{p+1}{\ell} \ell c_\ell z^{-\ell} \frac{\partial^{p+1-\ell} X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^{p+1-\ell}} \right) |_{y=z}. \end{aligned} \quad (10.5)$$

The right-hand-side of (10.5) is some expression in the α_k 's, here are a few of the fields $\frac{\partial^m X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^m} |_{y=z}$:

$$\begin{aligned} \frac{\partial X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y} |_{y=z} &= \frac{1}{n} \sum_k \alpha_k z^{-\frac{k+n}{n}} \\ \frac{\partial^2 X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^2} |_{y=z} &= \frac{1}{n^2} \sum_k (\alpha_k^{(2)} - (k+n)\alpha_k) z^{-\frac{k+2n}{n}} \\ \frac{\partial^3 X(y^{\frac{1}{n}}, z^{\frac{1}{n}})}{\partial y^3} |_{y=z} &= \frac{1}{n^3} \sum_k (\alpha_k^{(3)} - \frac{3}{2}(k+2n)\alpha_k^{(2)} + (k+n)(k+2n)\alpha_k) z^{-\frac{k+3n}{n}}, \end{aligned}$$

where $\alpha_k^{(2)} = \sum_j : \alpha_{-j} \alpha_{k+j} :$ and $\alpha_k^{(3)} = \sum_{i,j} : \alpha_{-i} \alpha_{-j} \alpha_{i+j+k} :$

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JOHAN VAN DE LEUR

FACULTY OF APPLIED MATHEMATICS

UNIVERSITY OF TWENTE

P.O. BOX 217

7500 AE ENSCHEDE

THE NETHERLANDS